Abstract Algebra I
Problem Set 5: Rings

Example style questions are marked with (*).

Rings

Question 1. Let R be a principal ideal domain, so, in particular, every element factorises uniquely up to ordering and multiplication by a unit. Prove that $x \in R$ is prime if and only if x is irreducible. (In fact, some rings can have unique factorisation without being PIDs. Such rings are called *unique factorisation domains* (UFDs))

Question 2. (*) Prove that every finite ring is partitioned by two sets, zero divisors and units. (Units are invertible elements of the ring.) Find a counterexample for infinite rings.

Question 3. Prove that a field can not have any zero divisors.

Question 4. (*) Consider a ring $(R, +, \cdot)$. Let 0 and 1 denote the additive and multiplicative identities of R. Prove that:

- if there exists a total order $>$ on R then $1 > 0$; and
- the group $(R, +) = \langle 1 \rangle$, that is, the additive group of R is generated by the multiplicative identity.

Question 5. (*) Show that

$$\begin{aligned} f :& \mathbb{Q}[x]/(x^2 - 2) \to \mathbb{Q}[\sqrt{2}] \\ :& f\big(a + b(x + (x^2 - 2))\big) = a + b\sqrt{2} \end{aligned}$$

is an isomorphism.

Homomorphisms

Question 6. Let 1_1 and 1_2 denote the multiplicative identities of two rings R_1 and R_2. Verify that a map $\varphi : R_1 \to R_2$ satisfying $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in R_1$ does *not* need to have $\varphi(1_1) = 1_2$ or even $1_2 \in \mathrm{Im}(\varphi)$.

Question 7. Prove that the set of endomorhpisms $\mathrm{End}(R)$ of a ring R is itself a ring with the same addition as R and function composition. Find the additive and multiplicative identities.

Ideals

Question 8. Find all ideals $\mathbb{Z}$ and in $\mathbb{Q}$. Prove your result.

Question 9. (*) Let $(F, +, \cdot)$ be a field, that is, $(F, \cdot)$ is an abelian group. Prove that the only ideals in F are $\{0\}$ and F itself, the trivial ideals.

Question 10. Show that the union of all cosets of an ideal I in R is equal to the set R.

Question 11. Let I be an ideal in R. Show that R/I is a ring with operations

$$\begin{aligned} +(r_1 + I, r_2 + I) &= (r_1 + r_2) + I, \\ \cdot\,(r_1 + I, r_2 + I) &= r_1 r_2 + I. \end{aligned}$$

Challenging

Question 12. Show that the ideals in $\mathbb{Z}_n$ are precisely $d\mathbb{Z}_n$ where d divides n.